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SOME TWO PARAMETER PROBLEMS FOR SINGULARLY PERTURBED EQUATIONS --ETC(U)
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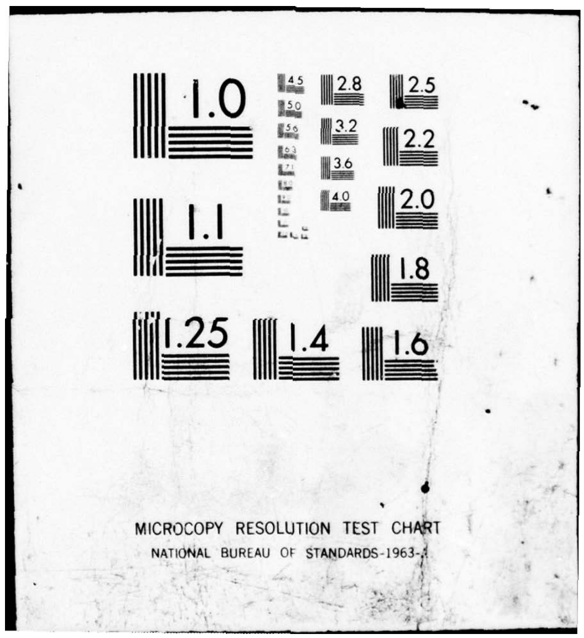
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(6) Some Two Parameter Problems for

Singularly Perturbed Equations of Elliptic Type.

by

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1. Introduction. We consider here some results on the existence and the asymptotic behavior of solutions of the singularly perturbed nonlinear problems

$$(1.1) \qquad \epsilon\Delta u = \mu A(x,u)\cdot \nabla u + h(x,u)$$

and

$$(1.2) \qquad \epsilon\Delta u = \mu \sum_{j=1}^{N} b_j(x,u)\, u_{x_j}^2 + h(x,u)$$

for x in a bounded region Ω in $\mathbb{R}^N$ $(N \geq 2)$ which satisfy along the boundary of Ω either Dirichlet - or Robin - type conditions. The parameters ϵ and μ are assumed throughout to be small and positive and to tend to zero in an interrelated fashion. We will determine precisely how the relative sizes of ϵ and μ influence the behavior of solutions of (1.1) and (1.2), especially near the boundary of Ω. At the same time we will give estimates on these solutions in regions where up to now complete information has been lacking. Such regions often have corners and/or boundaries which coincide with solutions of certain lower-order equations (that is, characteristic boundaries).

Problems of this type were first studied for linear differential equations in two dimensions (that is, $A(x,u) = (a(x,y), b(x,y))$ and $h(x,u) = h(x,y)u + g(x,y)$ in (1.1)) by O'Malley in a series of papers [16 - 18]. More recently, Butuzov [3,4] has considered this linear equation with Dirichlet boundary data in a rectangle in order to examine the interaction of the boundary with the characteristic curves of the "semi-reduced" equation $\mu(a(x,y), b(x,y))\cdot(u_x, u_y) + h(x,y)u + g(x,y) = 0$. Finally van Harten in his dissertation [20] and in [21] has discussed certain linear and nonlinear equations of the form (1.1) in dimensions greater than two when the parameter μ is a power of ϵ.

The article of O'Malley [18] the monograph of Lions [15] and the dissertation of van Harten [20] also contain interesting discussions of related singular perturbation problems as well as many additional references to the literature. In particular, O'Malley treats many results on two-parameter problems for ordinary differential equations and these can serve as motivation for some of the theorems given below. Our approach here is fashioned for the most part by our previous discussions of singularly perturbed partial differential equations of the form (1.1), (1.2) for the case $\mu = 1$ which are contained in [12,13]. These results and those of the present paper are proved using the comparison techniques of Amann [1] which compel us to restrict the right hand side of equation (1.2) to be at most a quadratic function of the first derivatives of u .

Part I

Dirichlet Problems

2. <u>The Problem</u> $(\mathcal{P}_1)$. We consider first the Dirichlet problem

$$(\mathcal{P}_1) \qquad \begin{aligned} &\epsilon \Delta u = \mu A(x,u)\cdot\nabla u + h(x,u)\,, \ x \ \text{ in } \ \Omega\,, \\ &u(x;\epsilon,\mu) = \varphi(x)\,, \ x \ \text{ on } \ \Gamma = \partial\Omega\,, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded region which is described by a function $F : \mathbb{R}^N \to \mathbb{R}^1$ in the sense that $\Omega = \{x : F(x) < 0\}$. Consequently the boundary Γ of Ω is simply $F^{-1}(0)$. The parameters ϵ and μ are assumed to be small and positive, and to tend to zero in an interrelated fashion.

We will study the behavior of solutions of $(\mathcal{P}_1)$ with the help of certain solutions of the reduced equation $h(x,u) = 0$ obtained by formally setting ϵ and μ equal to zero. In order to simplify the presentation we will assume (without great loss of generality; cf. Remark 2.1 below) that $h(x,0) \equiv 0$ in Ω . Thus our study concerns those solutions of $(\mathcal{P}_1)$ which are close to zero in Ω ,

but which change rapidly near Γ in order to satisfy the prescribed boundary conditions. This rapid change is termed boundary layer behavior, and it is a characteristic occurrence in many singular perturbation problems (cf., for example, [18]). Since we are looking specifically for such solutions of $(\mathcal{P}_1)$ we define now the closed domain in $\overline{\Omega} \times \mathbb{R}^1$ in which these solutions lie, namely $\mathcal{D} = \overline{\Omega} \times \{u : |u| \leq d(x)\}$, where $d > 0$ is a smooth function defined in $\overline{\Omega}$ such that $|\varphi(x)| \leq d(x) \leq |\varphi(x)| + \delta$ for x in $\Gamma_{\delta/2}$ and $d(x) \leq \delta$ for x in Γ_δ'. (Here $\delta > 0$ is a small constant and we introduce the sets $\Gamma_\rho = \{x \text{ in } \overline{\Omega} : \operatorname{dist}(x,\Gamma) \leq \rho\}$ and $\Gamma_\rho' = \{x \text{ in } \overline{\Omega} : \operatorname{dist}(x,\Gamma) \geq \rho\}$ for any positive constant ρ.)

In terms of the domain $\mathcal{D}$ we can now state the smoothness assumptions which we impose on the functions occurring in the statement of the problem. First of all, each of the components a_j of the vector function $A(x,u) = (a_1(x,u), \ldots, a_N(x,u))$ as well as the function h are assumed to be of Hölder class $C^{(0,\alpha)}$ $(0 < \alpha < 1)$ with respect to x and of class $C^{(1)}$ with respect to u in $\mathcal{D}$. Similarly, the functions F and φ are assumed to be of Hölder class $C^{(2,\alpha)}(\mathbb{R}^N)$ and $C^{(2,\alpha)}(\Gamma_\delta)$, respectively. Finally we assume that the trivial solution of the equation $h(x,u) = 0$ is stable in the sense that there exists a positive constant m such that $h_u(x,u) \geq m^2 > 0$ for (x,u) in $\mathcal{D}$; cf. [12,13].

We are now ready to discuss the existence of solutions of $(\mathcal{P}_1)$ and their behavior for small values of ϵ and μ. Suppose first that μ is asymptotically smaller than $\sqrt{\epsilon}$ $(\mu << \sqrt{\epsilon})$, that is, $\mu/\sqrt{\epsilon} \to 0^+$ as $\epsilon \to 0^+$. Then we expect that the influence of the term $\mu A \cdot \nabla u$ on the behavior of solutions will be negligible compared to that of h. Indeed, there is the following result.

<u>Theorem 2.1</u>. <u>Under the above smoothness and stability assumptions there exists a positive constant</u> ϵ_o <u>such that the problem</u> $(\mathcal{P}_1)$ <u>has a solution</u> $u = u(x;\epsilon,\mu)$ <u>of class</u> $C^{(2,\alpha)}(\bar{\Omega})$ <u>whenever</u> $0 < \epsilon \le \epsilon_o$ <u>and</u> $0 < \mu << \sqrt{\epsilon}$. <u>In addition, for</u> x <u>in</u> $\bar{\Omega}$ <u>we have that</u>

$$|u(x;\epsilon,\mu)| \le K \exp[m_1 F(x)/\sqrt{\epsilon}] ,$$

<u>where</u> $K = \max_{\Gamma} |\varphi(x)|$ <u>and</u> $0 < m_1 < m L^{-1}$ <u>with</u> $L = \max_{\bar{\Omega}} \|\nabla F(x)\|$. <u>(Here and below</u> $\|\cdot\|$ <u>denotes the usual Euclidean norm in</u> $\mathbb{R}^N$.)

<u>Proof</u>. To prove all of the theorems involving Dirichlet problems we will use the following well known result (cf. for example [1]). Namely, for $\mathcal{F}(x,u,\nabla u,\mu) = \mu A(x,u)\cdot\nabla u + h(x,u)$ or $\mu \sum_{j=1}^{N} b_j(x,u) u_{x_j}^2 + h(x,u)$ satisfying the given smoothness conditions in $\mathcal{D}$ the boundary value problem $\epsilon \Delta u = \mathcal{F}(x,u,\nabla u,\mu)$, x in Ω, $u = \varphi(x)$, x on $\Gamma = \partial\Omega$, has a solution $u = u(x;\epsilon,\mu)$ in $\mathcal{D}$ of class $C^{(2,\alpha)}(\bar{\Omega})$ for each $\epsilon > 0$ and $\mu > 0$ for which there exist functions $\underline{\omega},\bar{\omega}$ of class $C^{(2,\alpha)}(\bar{\Omega})$ such that $\underline{\omega} \le \bar{\omega}$, $\underline{\omega}|_{\Gamma} \le \varphi(x) \le \bar{\omega}|_{\Gamma}$, and $\epsilon\Delta\underline{\omega} \ge \mathcal{F}(x,\underline{\omega},\nabla\underline{\omega},\mu)$, $\epsilon\Delta\bar{\omega} \le \mathcal{F}(x,\bar{\omega},\nabla\bar{\omega},\mu)$ for x in Ω. Moreover, this solution satisfies $\underline{\omega}(x;\epsilon,\mu) \le u(x;\epsilon,\mu) \le \bar{\omega}(x;\epsilon,\mu)$ for x in $\bar{\Omega}$ and for all such $\epsilon > 0$ and $\mu > 0$. (The function $\underline{\omega}(\bar{\omega})$ is called a lower (upper) solution of the boundary value problem.) Such inequality techniques have been used by Dorr, Parter and Shampine [7] and the author [11] for related one-dimensional problems, and by Eckhaus and de Jager [9], Eckhaus [8] and the author [12,13] for higher dimensional ones.

Thus with regard to Theorem 2.1 we define for x in $\bar{\Omega}$ the functions $\underline{\omega},\bar{\omega}$ by $\underline{\omega}(x,\epsilon) = -\bar{\omega}(x,\epsilon) = -K\exp[m_1 F(x)/\sqrt{\epsilon}]$, where K and m_1 are given above. Clearly for x on Γ $\underline{\omega}(x,\epsilon) \le \varphi(x) \le \bar{\omega}(x,\epsilon)$, and it is just as easy to see that the differential inequalities are satisfied. For example, consider $\underline{\omega}$. (The verification that $\bar{\omega}$ satisfies the proper

inequality is similar and is omitted.) Then we have that (for $\mathfrak{J}(x,u,\nabla u,\mu) = \mu A(x,u)\cdot\nabla u + h(x,u)$)

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathfrak{J}(x,\underline{\omega},\nabla\underline{\omega},\mu) &= \{-m_1\Delta F\sqrt{\epsilon} - m_1^2\|\nabla F\|^2 \\ &\quad + (\mu/\sqrt{\epsilon})m_1(A\cdot\nabla F) + h(x,0) \\ &\quad + h_u(x,\xi)\}|\underline{\omega}| \\ &\geq \{-m_1|\Delta F|_\infty\sqrt{\epsilon} - m_1^2 L^2 \\ &\quad - (\mu/\sqrt{\epsilon})m_1|A\cdot\nabla F|_\infty + m^2\}|\underline{\omega}| \\ &\geq 0 \text{ by our choice of } m_1\end{aligned}$$

for ϵ and μ sufficiently small, say $0 < \epsilon \leq \epsilon_0$, since $\mu/\sqrt{\epsilon} \to 0^+$ as $\epsilon \to 0^+$. Here (x,ξ) in $\mathfrak{D}$ is the appropriate intermediate point. The conclusion of the theorem now follows from the differential inequality theorem just quoted.

Suppose next that the parameters μ and $\sqrt{\epsilon}$ are asymptotically comparable ($\mu \sim \sqrt{\epsilon}$), that is, there exist positive constants ℓ and ℓ_1 such that $0 < \ell_1 \leq \mu/\sqrt{\epsilon} \leq \ell$ as $\epsilon \to 0^+$. Then the term $\mu A\cdot\nabla u$ contributes more to the description of the behavior of solutions of $(\mathcal{P}_1)$ than it did in the previous case. The precise result is the next theorem.

Theorem 2.2. *Under the above smoothness and stability assumptions there exists a positive constant* ϵ_0 *such that the problem* $(\mathcal{P}_1)$ *has a solution* $u = u(x;\epsilon,\mu)$ *of class* $C^{(2,\alpha)}(\bar{\Omega})$ *whenever* $0 < \epsilon \leq \epsilon_0$ *and* $\mu \sim \sqrt{\epsilon}$. *In addition, for* x *in* $\bar{\Omega}$ *we have that*

$$|u(x;\epsilon,\mu)| \leq K\exp[m_2 F(x)/\sqrt{\epsilon}],$$

where $m_2 > 0$ is such that $m_2^2 L^2 + m_2 \ell L_1^2 < m^2$. Here K and L are as before and $L_1 = \max_{\mathcal{D}} | A(x,u) \cdot \nabla F(x) |$.

Proof. Define the same functions $\underline{\omega}, \overline{\omega}$ as in the proof of Theorem 2.1 (with m_1 replaced by m_2), and note that

$$\begin{aligned} \epsilon \Delta \underline{\omega} - \mathcal{F}(x, \underline{\omega}, \nabla \underline{\omega}, \mu) &\geq \{ -m_2 | \Delta F |_\infty \sqrt{\epsilon} - m_2^2 L^2 \\ &\quad - \ell m_2 L_1 + m^2 \} |\underline{\omega}| \\ &\geq 0 \text{ by our choice of } m_2 \end{aligned}$$

for ϵ sufficiently small, say $0 < \epsilon \leq \epsilon_0$.

Suppose now that μ is asymptotically larger than $\sqrt{\epsilon}$ ($\mu >> \sqrt{\epsilon}$), that is, $\sqrt{\epsilon}/\mu \to 0^+$ as $\mu \to 0^+$. Since μ is much larger than $\sqrt{\epsilon}$ we expect that the term $\mu A \cdot \nabla u$ will determine the behavior of solutions of $(\mathcal{P}_1)$ at least near the boundary of Ω. Consequently we must examine the behavior of the characteristic curves of the "semi-reduced" equation $(\mathcal{S})$ $\mu A(x,u) \cdot \nabla u + h(x,u) = 0$. Three distinct cases are present; cf. for example also the related discussion in [6]. First, all of the characteristic curves of $(\mathcal{S})$ may leave Ω along Γ either tangentially or nontangentially, that is,

$$(*) \quad A(x,u) \cdot \nabla F(x) \geq 0 \quad \text{for} \quad (x,u) \quad \text{in} \quad \mathcal{D}_\delta = \mathcal{D} \cap (\Gamma_\delta \times \mathbb{R}^1) .$$

(If Γ is itself a characteristic curve then $A(x,u) \cdot \nabla F(x) \equiv 0$ on Γ.) We have the following result.

Theorem 2.3. Under the above smoothness and stability assumptions there exists a positive constant μ_0 such that the problem $(\mathcal{P}_1)$ has a solution $u = u(x; \epsilon, \mu)$ whenever $0 < \mu \leq \mu_0$, $\sqrt{\epsilon} << \mu$ and the inequality $(*)$ obtains. In addition, for x in $\overline{\Omega}$ we have that

$$|u(x;\epsilon,\mu)| \le K\exp[m_1 F(x)/\sqrt{\epsilon}\,] + \rho(\epsilon,\mu) ,$$

where K and m_1 are as before, and $\rho(\epsilon,\mu)$ is a positive function of order $(\mu/\sqrt{\epsilon}\,)\exp[-\delta/\sqrt{\epsilon}\,] << 1$.

Proof. Define for x in $\bar{\Omega}$ the functions $\underline{\omega}, \bar{\omega}$ by $\underline{\omega}(x;\epsilon,\mu) = -\bar{\omega}(x;\epsilon,\mu) = -(K\exp[m_1 F(x)/\sqrt{\epsilon}\,] + \rho(\epsilon,\mu))$, where $\rho(\epsilon,\mu) = m^{-2}\rho_1(\epsilon,\mu)$ for $\rho_1(\epsilon,\mu) =$

$$\{(\mu/\sqrt{\epsilon}\,)m_1|A\cdot\nabla F|_\infty + m_1|\Delta F|_\infty\sqrt{\epsilon} + m_1^2 L^2\}\exp[-m_1\delta/\sqrt{\epsilon}\,] .$$

Then proceeding as in the proof of Theorem 2.1 we have that

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathfrak{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) \ge \{&-m_1|\Delta F|_\infty\sqrt{\epsilon} - m_1^2 L^2 \\ &+ (\mu/\sqrt{\epsilon}\,)m_1(A\cdot\nabla F) + m^2\}|\underline{\omega}+\rho| \\ &+ \rho_1(\epsilon,\mu) .\end{aligned}$$

Now for x in Γ_δ $A\cdot\nabla F \ge 0$ and so we have the desired inequality since $m_1 < mL^{-1}$. But for x in Γ_δ' the term $\{\cdot\}|\underline{\omega}+\rho|$ is such that $|\{\cdot\}|\,|\underline{\omega}+\rho| \le \rho_1(\epsilon,\mu)$ and so the desired inequality follows here also.

Suppose next that all of the characteristic curves of $(\mathcal{S})$ leave Ω along Γ nontangentially, that is, there exists a positive constant k such that

$$(**)\quad A(x,u)\cdot\nabla F(x) \ge k\|\nabla F(x)\|^2 \quad \text{for} \quad (x,u) \text{ in } \mathcal{D}_\delta .$$

Then the conclusion of Theorem 2.3 can be strengthened as follows.

Theorem 2.4. Make the same assumptions as in Theorem 2.3 with the condition (*) replaced by (**) . Then the conclusion of Theorem 2.3 is valid with the argument of the exponential replaced by $kF(x)\mu/\epsilon$ and with $\rho(\epsilon,\mu) > 0$ of order $(\mu^2/\epsilon)\exp[-k\delta\mu/\epsilon] << 1$.

Note that since $\sqrt{\epsilon} << \mu$, $\epsilon/\mu << \sqrt{\epsilon}$, and so the boundary layer estimate of this theorem is sharper than that of Theorem 2.3.

Proof. Define for x in $\bar{\Omega}$ the functions $\underline{\omega}, \bar{\omega}$ by $\underline{\omega}(x; \epsilon, \mu) = -\bar{\omega}(x; \epsilon, \mu) = -(K \exp[k F(x) \mu/\epsilon] + \rho(\epsilon, \mu))$, where $\rho(\epsilon, \mu) = m^{-2} \rho_1(\epsilon, \mu)$ for $\rho_1(\epsilon, \mu) = \{\mu k | \Delta F|_\infty + (\mu^2/\epsilon)(k^2 \|\nabla F\|_\infty^2 + k |A \cdot \nabla F|_\infty)\} K \exp[-k \delta \mu/\epsilon]$. It follows that

$$\begin{aligned}\epsilon \Delta \underline{\omega} - \mathfrak{F}(x, \underline{\omega}, \nabla \underline{\omega}, \mu) \geq \{ & -k |\Delta F|_\infty \mu \\ & + (\mu^2/\epsilon)(-k^2 \|\nabla F\|^2 + k A \cdot \nabla F) \\ & + m^2 \} |\underline{\omega} + \rho| + \rho_1(\epsilon, \mu) .\end{aligned}$$

Now for x in Γ_δ $A \cdot \nabla F \geq k \|\nabla F\|^2$ and so the desired inequality is valid for μ sufficiently small, say $0 < \mu \leq \mu_0$. And for x in Γ_δ' the term $\{\cdot\} |\underline{\omega} + \rho|$ is bounded in absolute value by ρ_1; consequently, we have the desired inequality for all x in Ω.

Suppose finally that neither of the two previous cases obtains, that is, $A(x,u) \cdot \nabla F(x) \not\geq 0$ for (x,u) in $\mathfrak{D}_\delta$. Geometrically this means that some or all of the characteristic curves of $(\mathcal{S})$ are entering Ω along portions of Γ. Roughly speaking, in this case we have no control over the boundary layer behavior of solutions along portions of Γ where $A \cdot \nabla F$ is negative. This is reflected in the relatively crude estimate in the following theorem.

Theorem 2.5. Under the above smoothness and stability assumptions there exists a positive constant μ_0 such that the problem (η_1) has a solution $u = u(x; \epsilon, \mu)$ of class $C^{(2,\alpha)}(\bar{\Omega})$ whenever $0 < \mu \leq \mu_0$ and $\sqrt{\epsilon} << \mu$. In addition, if $A(x,u) \cdot \nabla F(x) \not\geq 0$ in $\mathfrak{D}_\delta$ then for x in $\bar{\Omega}$ we have the estimate

$$|u(x;\epsilon,\mu)| \le K \exp[m_3 F(x)/\mu] ,$$

<u>where</u> $m_3 > 0$ <u>is such that</u> $m_3 < m^2/L_2$ <u>with</u> $L_2 = \max_{\mathcal{D}} |A(x,u) \cdot \nabla F(x)|$ <u>and</u> K <u>as above</u>.

<u>Proof</u>. Define for x in $\bar{\Omega}$ the functions $\underline{\omega}, \bar{\omega}$ by $\underline{\omega}(x,\mu) = -\bar{\omega}(x,\mu) = -K \exp[m_3 F(x)/\mu]$. Then we have that

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathfrak{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) &\ge \{-m_3 |\Delta F|_\infty \epsilon/\mu \\ &\quad -m_3^2 \|\nabla F\|_\infty^2 \epsilon/\mu^2 \\ &\quad -m_3 L_2 + m^2\} |\underline{\omega}| \\ &\ge 0 \text{ by our choice of } m_3\end{aligned}$$

if μ is sufficiently small, say $0 < \mu \le \mu_0$, since $\epsilon << \sqrt{\epsilon} << \mu$.

It is possible to improve the estimate of Theorem 2.5 by considering more closely how the characteristics of $(\mathcal{S})$ interact with the boundary of Ω . Before doing this in the next section we make several remarks.

<u>Remark 2.1</u>. If $u = u_0(x) \not\equiv 0$ is a smooth solution of $h(x,u) = 0$ satisfying $h_u(x,u) \ge m^2 > 0$ for (x,u) in $\mathcal{D}(u_0) = \bar{\Omega} \times \{u : |u - u_0(x)| \le d_1(x)\}$ (where $d_1 > 0$ is a smooth function such that $|\varphi(x) - u_0(x)| \le d_1(x) \le |\varphi(x) - u_0(x)| + \delta$ for x in $\Gamma_{\delta/2}$ and $d_1(x) \le \delta$ for x in Γ'_δ) then the above theory applies with minor modifications. This can be seen by considering the problem (η_1) for the function $z = u - u_0(x)$.

Similarly, if the operator $\epsilon\Delta$ is replaced by ϵL where L is the general linear second-order uniformly elliptic operator in N variables then our results for this more general problem are qualitatively the same as those for (η_1) .

<u>Remark 2.2</u>. The stability assumption that $(\dagger)$ $h_u(x,u) \geq m^2 > 0$ for (x,u) in $\mathscr{D}$ is only one of several related assumptions which often have to be made in order to study the problems in this paper. The reader can consult [12] for precise statements of these assumptions. With regard to the condition $(\dagger)$ we note that it is sometimes necessary in order to prove results like Theorems 2.1 - 2.3 and 2.5 to make the weaker assumption that

(i) $h_u(x,0) \geq m^2 > 0$ in $\bar{\Omega}$; and

(ii) $\varphi(x) \int_0^{\eta} h(x,s)\,ds > 0$ for x on Γ and

$$0 < \eta \leq \varphi(x) \text{ (or } \varphi(x) \leq \eta < 0 \text{) .}$$

Such restrictions allow larger "boundary layer jumps" $|\varphi(x)|$; cf., for example, [10], [12] or Example (E2') below.

<u>Remark 2.3</u>. Some of the results of this section and several extensions are already known. The case of a two-dimensional linear Dirichlet problem $(\mathcal{P}_1)$, that is, $A(x,u) = (a(x,y), b(x,y))$ and $h(x,u) = h(x,y)u + g(x,y)$, was studied in [16,18], [3,4] and [20]. In addition, the nonlinear problem $(\mathcal{P}_1)$ with Ω the N-dimensional unit ball and $\mu = \sqrt{\epsilon}$ was considered recently in [21]. These papers treat the problem of constructing the complete asymptotic expansion of the solution, a problem which we have not addressed here.

<u>Remark 2.4</u>. If $A(x,u) \cdot \nabla F(x) > 0$ for (x,u) in $\mathscr{D}_\delta$ then a compactness argument shows the existence of a positive constant k such that $(**)$ $A(x,u) \cdot \nabla F(x) \geq k \|\nabla F(x)\|^2$ in $\mathscr{D}_\delta$. Similarly, if there is a point x_o on Γ such that $\nabla F(x_o) = 0$, $A(x_o, u) \neq 0$ and $A(x,u) \cdot \nabla F(x) > 0$ (for all u of interest) in a punctured neighborhood G of x_o then the inequality $(**)$ obtains in $G \cup \{x_o\}$. Thus, with regard to Theorem 2.4, the boundary of Ω is allowed to have finitely many corners, that is, points at which ∇F vanishes.

<u>Remark 2.5</u>. The following idea may have occurred to the reader in the case that $\sqrt{\epsilon} << \mu$. Namely suppose one first solves the "semi-reduced" problem consisting of the equation $(\mathcal{S})$ together with a boundary condition along a portion of Γ and then uses this solution to approximate the solution of $(\mathcal{N}_1)$. Specifically the semi-reduced problem is

$$(\mathcal{S}\mathcal{P}) \qquad \begin{aligned} &\mu A(x,u)\cdot\nabla u + h(x,u) = 0\,, \quad x \text{ in } \Omega\,, \\ &u(x,\mu) = \varphi(x)\,, \quad x \text{ on } \Gamma_1 \subset \Gamma\,, \end{aligned}$$

where Γ_1 is the portion of Γ on which $A(x,u)\cdot\nabla F(x) < 0$ for all u of interest. A sufficient condition for the solvability of $(\mathcal{S}\mathcal{P})$ is that $h(x,0) \equiv 0$ and $h_u(x,u) \geq m^2 > 0$ for (x,u) in $\mathcal{D}$; cf. [14] for a proof in the case $N = 2$. Then the solution $u = u_o(x,\mu)$ of $(\mathcal{S}\mathcal{P})$ satisfies in $\overline{\Omega}$

$$u_o(x,\mu) = \mathcal{O}(K_1 \exp[mF_1(x)/\mu])\,,$$

where $K_1 = \max_{\Gamma_1} |\varphi(x)|$ and $F_1 = F|_{\Gamma_1}$. Consequently, along the portion of Γ where $A(x,u)\cdot\nabla F(x) < 0$ an easy estimation shows that the solution u of $(\mathcal{N}_1)$ satisfies $|u - u_o|_\infty = o(1)$ as $\mu \to 0^+$, a result which agrees with the estimate of Theorem 2.5. On the other hand, along the portion of Γ where $A(x,u)\cdot\nabla F(x) \geq 0$ the same kind of argument used in the proofs of Theorems 2.1 - 2.3 shows that $|u - u_o| = \mathcal{O}(K\exp[F(x)/\sqrt{\epsilon}])$. But for such x, $u_o = o(1)$ as $\mu \to 0^+$ and so this estimate agrees with our previous one. All of this shows that we can obtain the results of this section in the case that $\sqrt{\epsilon} << \mu$ by using instead a solution of the semi-reduced problem $(\mathcal{S}\mathcal{P})$. We find it more convenient to use the solution of the "totally" reduced problem $h(x,u) = 0$.

<u>3. Some Extensions</u>. We consider in this section some sharper results than Theorem 2.5 in the case that $\sqrt{\epsilon} \ll \mu$ and $A(x,u)\cdot\nabla F(x) \not> 0$ along certain portions of Γ. Specifically we have in mind the situation where a characteristic curve of the semi-reduced equation $(\mathcal{S})$ is tangent to an isolated noncorner point of Γ.

Suppose then that x_o on Γ is such a point, that is, $\nabla F(x_o) \neq 0$, $A(x_o, u) \neq 0$ but $A(x_o, u)\cdot\nabla F(x_o) = 0$ for all u of interest. Let $\gamma(x,u) = A(x,u)\cdot\nabla F(x)$ in a neighborhood G of x_o defined by $G = \{x : g(x) < 0\}$. In terms of γ and G, we assume that $\gamma(x_o, u) = 0$ and that γ changes sign in passing through x_o, that is,
$\gamma(x,u) < 0$ in $G^-\setminus\{x_o\}$ where $G^- = G \cap \{x : A(x,u)\cdot\nabla g < 0\}$ and
$\gamma(x,u) > 0$ in $G^+\setminus\{x_o\}$ where $G^+ = G \cap \{x : A(x,u)\cdot\nabla g > 0\}$.

Proceeding as in the previous section we now examine the solutions of $(\mathcal{P}_1)$ in the subsets $G^\pm$ of G. For simplicity we assume that the point of tangency is $x_o = 0$.

<u>Lemma 3.1</u>. <u>Suppose that</u> $u = u(x;\epsilon,\mu)$ <u>is the solution of the problem</u> $(\mathcal{P}_1)$ <u>in the domain</u> $\mathcal{D}$. <u>Then in the subset</u> $\overline{G^+}$ <u>of</u> G <u>where</u> $\gamma \geq 0$ <u>we have that</u>

$$u(x;\epsilon,\mu) = \mathcal{O}(K_1 \exp[kF(x)/\nu]) ,$$

<u>where</u> $K_1 = \max\limits_{G^+\cap\Gamma} |\varphi(x)|$ <u>and</u> $\nu = \sqrt{\epsilon}$ <u>for those</u> x <u>such that</u> $\gamma = \mathcal{O}(\sqrt{\epsilon}/\mu)$ <u>while</u> $\nu = \mathcal{O}(\epsilon/(\gamma\mu))$ <u>for those</u> x <u>such that</u> $\gamma \gg \sqrt{\epsilon}/\mu$. <u>Here</u> $k > 0$ <u>is a known constant depending on</u> γ.

Note that for $\gamma = 0$ ($\gamma = \mathcal{O}(1)$, $\gamma > 0$) this estimate agrees with the one of Theorem 2.3 (Theorem 2.4) but it provides much more information than these results.

Proof. To prove the lemma it is sufficient to note that for x in $\overline{G^+ \cap \Omega}$ the function $\omega(x,\nu) = K_1 \exp[kF(x)/\nu]$ with k and ν appropriately chosen is a barrier function for $(\mathcal{P}_1)$. That is, ω is an upper solution and $-\omega$ is a lower solution of $(\mathcal{P}_1)$; cf. [5] or [9]. This follows in the case of $\underline{\omega} = -\omega$, for example, because

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathfrak{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) &\geq \{-k|\Delta F|_\infty \epsilon/\nu \\ &\quad -k^2\|\nabla F\|^2\epsilon/\nu^2 \\ &\quad +k\gamma(x,\underline{\omega})\mu/\nu + m^2\}\omega \\ &\geq 0 \text{ for } k \text{ appropriately chosen}\end{aligned}$$

provided $\nu = \sqrt{\epsilon}$ for all x such that $\gamma(x,\underline{\omega}) = \mathcal{O}(\sqrt{\epsilon}/\mu)$ and $\nu = \mathcal{O}(\epsilon/(\gamma\mu))$ for all x such that $\sqrt{\epsilon}/\mu << \gamma(x,\underline{\omega})$. A similar argument shows that ω is an upper solution, and so $|u(x;\epsilon,\mu)| \leq \omega(x,\nu)$ in $\overline{G^+ \cap \Omega}$.

Consider finally the subset G^- of G where $\gamma < 0$.

Lemma 3.2. Suppose that $u = u(x;\epsilon,\mu)$ is the solution of $(\mathcal{P}_1)$ in the domain $\mathfrak{D}$. Then in the subset $\overline{G^-}$ of $\overline{G}$ where $\gamma \leq 0$ we have that

$$u(x;\epsilon,\mu) = \mathcal{O}(K_1 \exp[kF(x)/\nu]),$$

where K_1 and k are as before and $\nu = \sqrt{\epsilon}$ for those x such that $|\gamma| = \mathcal{O}(\sqrt{\epsilon}/\mu)$ while $\nu = \mathcal{O}(|\gamma|\mu)$ for those x such that $|\gamma| >> \mathcal{O}(\sqrt{\epsilon}/\mu)$.

Again note that for $|\gamma|$ sufficiently small ($|\gamma| = \mathcal{O}(1)$, $\gamma < 0$) this estimate agrees with that of Theorem 2.1 (Theorem 2.5). However, it provides more precise information than these results.

Proof. The proof follows that of Lemma 3.1 once we note that the "γ-term" in the corresponding inequalities is harmful when $|\gamma(x, \pm\omega)| \gg \mathcal{O}(\sqrt{\epsilon}/\mu)$ and so we must choose $\nu = \mathcal{O}(|\gamma|\mu)$ for such x.

Before considering another class of Dirichlet problems in the next section we make some remarks about these lemmas.

Remark 3.1. The estimates of Lemmas 3.1 and 3.2 give the following qualitative picture of the behavior of the solution near the point of tangency $x_o = 0$. Namely, for $\gamma(x, \pm\omega) \sim \pm\|x\|$, at the outer edge of G^- the boundary layer is quite thick (of order μ) but decreases like $\mathcal{O}(\|x\|\mu)$ as $\|x\|$ decreases until $\|x\| = \mathcal{O}(\sqrt{\epsilon}/\mu)$ after which the boundary layer thickness is of order $\sqrt{\epsilon}$. The boundary layer thickness remains of order $\sqrt{\epsilon}$ after x passes through $x_o = 0$ into G^+ until $\|x\| \gg \sqrt{\epsilon}/\mu$ after which the thickness decreases to $\mathcal{O}(\epsilon/\|x\|\mu)$ as $\|x\|$ increases. Finally the thickness shrinks to order ϵ/μ as $\gamma > 0$ increases to order one.

The linear equation $\epsilon(u_{xx} + u_{yy}) = \mu u_y + u$, for (x,y) in the unit disk centered at $(1,0)$, serves as a simple illustration of this remark. Here $\gamma(x,y,u) \equiv y$ and the points of tangency are $(0,0)$ and $(2,0)$.

Remark 3.2. In Lemmas 3.1 and 3.2 we studied the solution of $(\mathcal{P}_1)$ in the domain $\mathcal{D}$. This is justified by our stability assumption which guarantees the uniqueness of the solution of $(\mathcal{P}_1)$ in the domain $\mathcal{D}$; cf. [19].

Remark 3.3. The ideas of this section can be extended to cover cases in which a portion of Γ actually coincides with a characteristic curve of $(\mathcal{S})$ and/or in which Γ has corners which interact with the characteristics.

Both situations would arise if, for example, Ω were a rectangle in $\mathbb{R}^2$ and the characteristics were horizontal or vertical lines. We discuss such a problem in Section 7.

Remark 3.4. We note finally that if $\gamma < 0$ (in G^-) and $\mu = 1$ then Lemma 3.2 shows that the only way to have a close approximation to the solution of $(\mathcal{P}_1)$ in G^- is to make K_1 zero. That is, we must choose a solution $u = u_o(x)$ of the reduced equation

$$(\mathcal{S}_1) \qquad A(x,u) \cdot \nabla u + h(x,u) = 0$$

which satisfies the prescribed boundary data along that portion of Γ. It is this solution that we use to approximate the solution of $(\mathcal{P}_1)$ and to which we would add a boundary layer corrector term along the part of Γ where u_o does not necessarily satisfy the boundary data ; cf. for example [9], [18; Sec. 7] and [15]. This remark also makes sense geometrically because along that part of Γ where $\gamma < 0$ the characteristic curves of $(\mathcal{S}_1)$ are incoming and so it is possible, in general, to find a solution of $(\mathcal{S}_1)$ assuming the prescribed boundary values there.

4. The Problem $(\mathcal{P}_2)$. We consider next the following problem whose right hand side contains a quadratic nonlinearity, namely

$$(\mathcal{P}_2) \qquad \begin{aligned} &\epsilon \Delta u = \mu \sum_{j=1}^{N} b_j(x,u)\, u_{x_j}^2 + h(x,u) , \; x \text{ in } \Omega , \\ &u(x ; \epsilon , \mu) = \varphi(x) , \; x \text{ on } \Gamma = \partial\Omega . \end{aligned}$$

The functions b_j are assumed to have the same smoothness properties as the functions a_j considered in Section 2, and the remaining functions and sets are as before. We will assume again that $h(x,0) \equiv 0$ in Ω and that the

trivial solution is stable in the sense that there exists a positive constant m such that $h_u(x,u) \geq m^2 > 0$ in the same domain $\mathcal{D}$.

Our discussion of the asymptotic behavior of solutions of (η_2) parallels that of Section 2 for (η_1); however, the quadratic nonlinearity introduces some distinctive changes, as will become apparent shortly.

Suppose first that μ is asymptotically smaller than $\epsilon (\mu << \epsilon)$, that is, $\mu/\epsilon \to 0^+$ as $\epsilon \to 0^+$. The quadratic dependence of the right hand side on the first derivatives of u prompts us to compare μ and ϵ instead of μ and $\sqrt{\epsilon}$; cf. the following proof of Theorem 4.1. Then the analog of Theorem 2.1 is the following result.

Theorem 4.1. Under the above smoothness and stability assumptions there exists a positive constant ϵ_0 such that the problem (η_2) has a solution $u = u(x;\epsilon,\mu)$ of class $C^{(2,\alpha)}(\overline{\Omega})$ whenever $0 < \epsilon \leq \epsilon_0$ and $0 < \mu << \epsilon$. In addition, for x in $\overline{\Omega}$ we have that

$$|u(x;\epsilon,\mu)| \leq K \exp[m_1 F(x)/\sqrt{\epsilon}],$$

where $K = \max_{\Gamma} |\varphi(x)|$ and $0 < m_1 < m L^{-1}$ with $L = \max_{\overline{\Omega}} \|\nabla F(x)\|$.

Proof. Simply proceed as in the proof of Theorem 2.1 with $\mathcal{F}(x,u,\nabla u,\mu) = \mu \sum_{j=1}^{N} b_j(x,u) u_{x_j}^2 + h(x,u)$ and $\underline{\omega}, \overline{\omega}$ as defined there. (Note that

$$\mu \sum_{j=1}^{N} b_j(x,\omega)\omega_{x_j}^2 = \mathcal{O}((\mu/\epsilon)\exp[2m_1 F(x)/\sqrt{\epsilon}]) = \mathcal{O}(\mu/\epsilon) \to 0 \text{ as } \epsilon \to 0^+$$

for $\omega = \underline{\omega}, \overline{\omega}$.)

Next, if μ is asymptotically equal to ϵ $(\mu \sim \epsilon)$, that is, if there exist positive constants ℓ and ℓ_1 such that $0 < \ell_1 \leq \mu/\epsilon \leq \ell$, then we expect that a result like Theorem 4.1 will obtain with m_1 replaced by another suitable constant. The precise result is the next theorem whose proof mimics that of Theorem 2.2 and is therefore omitted.

Theorem 4.2. Under the above smoothness and stability assumptions there exists a positive constant ϵ_0 such that the problem $(\mathcal{P}_2)$ has a solution $u = (x; \epsilon, \mu)$ of class $C^{(2,\alpha)}(\overline{\Omega})$ whenever $0 < \epsilon \leq \epsilon_0$ and $0 < \ell_1 < \mu/\epsilon \leq \ell$. In addition, for x in $\overline{\Omega}$ we have that

$$|u(x; \epsilon, \mu)| \leq K \exp[m_2 F(x)/\sqrt{\epsilon}] ,$$

where $0 < m_2 < m(L^2 + \ell K L_1)^{-1/2}$ with K and L as above, and

$$L_1 = \max_{\mathcal{D}} \left| \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x) \right| .$$

Suppose finally that μ is asymptotically larger than $\epsilon(\mu >> \epsilon)$, that is, $\frac{\epsilon}{\mu} \to 0^+$ as $\mu \to 0^+$. Then the effect of the quadratic terms becomes really noticeable for the first time, and we must exercise some care. As with the analogous situation in our discussion of $(\mathcal{P}_1)$ three distinct cases present themselves. Suppose first that

$$(*) \qquad \varphi(x) \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x) \geq 0 \quad \text{for} \quad (x,u) \quad \text{in} \quad \mathcal{D}_\delta .$$

Geometrically this means that the solution $(\mathcal{P}_2)$ must be convex $(\varphi \geq 0)$ or concave $(\varphi \leq 0)$ near Γ, since for such x $\Delta u \sim (\mu/\epsilon) \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x)$ has the sign of $\varphi(x)$ as a result of $(*)$.

Theorem 4.3. Under the above smoothness and stability assumptions there exists a positive constant μ_0 such that the problem $(\mathcal{P}_2)$ has a solution $u = u(x; \epsilon, \mu)$ of class $C^{(2,\alpha)}(\overline{\Omega})$ whenever $0 < \mu \leq \mu_0$, $0 < \epsilon << \mu$ and the inequality $(*)$ obtains. In addition, for x in $\overline{\Omega}$ we have that

$$-K \exp[m_1 F(x)/\sqrt{\epsilon}] \leq u(x; \epsilon, \mu) \leq 0 \quad (\text{if} \quad \varphi \leq 0)$$

or

$$0 \leq u(x; \epsilon, \mu) \leq K \exp[m_1 F(x)/\sqrt{\epsilon}] \quad (\text{if} \quad \varphi \geq 0) ,$$

where K and m_1 are as before.

<u>Proof</u>. Suppose for example that $\varphi \leq 0$ and define for x in $\overline{\Omega}$ $\overline{\omega} \equiv 0$ and $\underline{\omega}(x,\epsilon) = -K\exp[m_1 F(x)/\sqrt{\epsilon}\,]$. Clearly $\overline{\omega}$ is an upper solution of $(\mathcal{P}_2)$ and it follows just as easily that $\underline{\omega}$ is a lower solution. Indeed we have that (for $\mathfrak{F}(x,u,\nabla u,\mu) = \mu \sum_{j=1}^{N} b_j(x,u)\, u_{x_j}^2 + h(x,u)$)

$$\epsilon\Delta\underline{\omega} - \mathfrak{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) \geq \{-m_1 \,|\,\Delta F\,|_\infty \sqrt{\epsilon} - m_1^2 L^2$$

$$+ m_1^2 (\mu/\epsilon)\Big(\sum_{j=1}^{N} b_j(x,\underline{\omega})\, F_{x_j}^2\Big)\underline{\omega} + m^2\}\,|\underline{\omega}|$$

$$\geq 0 \text{ by our choice of } m_1$$

if μ is sufficiently small, say $0 < \mu \leq \mu_0$. This follows because for x in Γ_δ $-\sum_{j=1}^{N} b_j(x,\underline{\omega})\, F_{x_j}^2(x) \geq 0$ by assumption, while for x in Γ_δ' $|\underline{\omega}|$ is transcendentally small, that is, $|\underline{\omega}| = \mathcal{O}(\epsilon^n)$ for all $n \geq 1$.

Thus the conclusion of Theorem 4.3 in the case that $\varphi \leq 0$ follows from the differential inequality theorem quoted in the proof of Theorem 2.1. The case $\varphi \geq 0$ is handled similarly with $\underline{\omega} \equiv 0$ and $\overline{\omega}(x,\epsilon) = K\exp[\,m_1 F(x)/\sqrt{\epsilon}\,]$.

Suppose now that the inequality (*) holds in the stronger sense that there exists a positive constant k such that for (x,u) in $\mathcal{D}_\delta$

$$(**)\qquad \begin{aligned} &\sum_{j=1}^{N} b_j(x,u)\, F_{x_j}^2(x) \leq -k\,\|\nabla F(x)\|^2 \quad (\text{if } \varphi \leq 0) \text{ or} \\ &\sum_{j=1}^{N} b_j(x,u)\, F_{x_j}^2(x) \geq k\,\|\nabla F(x)\|^2 \quad (\text{if } \varphi \geq 0). \end{aligned}$$

We expect that the estimate of Theorem 4.3 can be sharpened accordingly. This is the content of the next theorem.

Theorem 4.4. *Under the above smoothness and stability assumptions there exists a positive constant* μ_o *such that the problem* $(\mathcal{P}_2)$ *has a solution* $u = u(x;\epsilon,\mu)$ *of class* $C^{(2,\alpha)}(\bar{\Omega})$ *whenever* $0 < \mu \le \mu_o$, $0 < \epsilon << \mu$, *the inequality* (**) *obtains and* $\|\nabla F(x)\| > 0$ *for* x *in* Γ_δ. *In addition, for* x *in* $\bar{\Omega}$ *we have that*

$$\nu k^{-1} K \ln[-F_o(x) + e^{-k_1 \nu^{-1}}] \le u(x;\epsilon,\mu) \le 0 \quad (\text{if } \varphi \le 0)$$

or

$$0 \le u(x;\epsilon,\mu) \le -\nu k_1^{-1} K \ln[-F_o(x) + e^{-k_1 \nu^{-1}}] \quad (\text{if } \varphi \ge 0),$$

where $\nu = \epsilon/\mu$, $F_o(x) = F(x)/(\max_{\Omega} |F(x)| + 1)$ *and* $0 < k_1 < k K$ *with* K *as above*.

Proof. Suppose again that $\varphi \le 0$ and define for x in $\bar{\Omega}$ $\bar{\omega} \equiv 0$ and

$\underline{\omega}(x,\nu) = \nu k_1^{-1} K \ln G(x,\nu)$, where $G(x,\nu) = -F_o(x) + e^{-k_1 \nu^{-1}}$. Clearly $\bar{\omega}$ is an upper solution and as for $\underline{\omega}$, we have that

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathcal{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) \ge -\{ & |\Delta F_o|_\infty \nu\epsilon G^{-1} \\ & + G^{-2} \|\nabla F_o\|^2 \\ & + k_1^{-1} K \mu\nu^2 (\sum_{j=1}^{N} b_j(x,\underline{\omega}) F_{o,x_j}^2) G^{-2} \\ & + m^2 \nu \ln G\} k_1^{-1} K .\end{aligned}$$

Now, for x in Γ_δ $-\sum_{j=1}^{N} b_j(x,\underline{\omega}) F_{o,x_j}^2 \ge k \|\nabla F_o\|^2$ by assumption and so the desired inequality follows by our choice of k_1 if μ is sufficiently small, say $0 < \mu \le \mu_o$. Finally, for x in Γ_δ' this inequality also obtains because the positive term $-m^2 \nu \ln G$ dominates all of the others.

Suppose finally that the functions b_j, F and/or φ are such that the inequality (*) does not obtain, that is, $\varphi(x) \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x) \not\geq 0$ in $\mathcal{D}_\delta$. Then as in the analogous situation for the problem $(\mathcal{P}_1)$ we have no control over the behavior of the solution of $(\mathcal{P}_2)$ near Γ. The most general statement we can make about this case is contained in the next theorem.

Theorem 4.5. Under the above smoothness and stability assumptions there exists a positive constant μ_0 such that the problem $(\mathcal{P}_2)$ has a solution $u = u(x; \epsilon, \mu)$ of class $C^{(2,\alpha)}(\overline{\Omega})$ whenever $0 < \mu \leq \mu_0$ and $0 < \epsilon << \mu$. In addition, if $\varphi(x) \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x) \not\geq 0$ in $\mathcal{D}_\delta$, then for x in $\overline{\Omega}$ we have that

$$|u(x; \epsilon, \mu)| \leq K \exp[m_2 F(x)/\sqrt{\mu}],$$

where $0 < m_2 < m (K L_1)^{-1/2}$ with K and L_1 as above.

Proof. Define for x in $\overline{\Omega}$ the functions $\underline{\omega}, \overline{\omega}$ by $\underline{\omega}(x,\mu) = -\overline{\omega}(x,\mu) = -K \exp[m_2 F(x)/\sqrt{\mu}]$. Then in the case of $\underline{\omega}$ we have that

$$\begin{aligned}\epsilon \Delta \underline{\omega} - \mathcal{F}(x, \underline{\omega}, \nabla \underline{\omega}, \mu) \geq \{ & -m_2 \, |\Delta F|_\infty \epsilon/\sqrt{\mu} \\ & - m_2^2 \, \|\nabla F\|_\infty^2 \, \epsilon/\mu \\ & + m_2^2 \Big(\sum_{j=1}^{N} b_j(x,\underline{\omega}) F_{x_j}^2 \Big) \underline{\omega} \\ & + m^2 \} \, |\underline{\omega}| \\ \geq 0 & \text{ by our choice of } m_2\end{aligned}$$

if μ is sufficiently small, say $0 < \mu \leq \mu_0$. Similarly $\overline{\omega}$ satisfies the opposite inequality, and so the theorem follows.

We close this section with several remarks.

Remark 4.1. It is possible to consider a more general problem than $(\mathcal{P}_2)$, namely

$$(\mathcal{P}_2') \quad \begin{aligned} &\epsilon \Delta u = \mu_1 \sum_{j=1}^{N} b_j(x,u) u_{x_j}^2 + \mu_2 A(x,u)\cdot \nabla u + h(x,u), \quad x \text{ in } \Omega , \\ &u(x;\epsilon,\mu_1,\mu_2) = \varphi(x), \quad x \text{ on } \Gamma , \end{aligned}$$

where ϵ, μ_1 and μ_2 are small interrelated parameters. However we have studied only the simpler problem $(\mathcal{P}_2)$ in order to isolate the effects of the quadratic nonlinearities on the asymptotic behavior of solutions. Depending on the relative sizes of ϵ, μ_1 and μ_2 solutions of $(\mathcal{P}_2')$ behave roughly like the solutions of $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ studied above.

Remark 4.2. In order to prove Theorem 4.4 we had to make the assumption that $\nabla F(x) \neq 0$ for x in Γ_δ, that is, we did not allow the boundary to have corners. If however $\nabla F(x_0) = 0$ at an isolated point x_0 on Γ then the conclusion of the theorem is still valid provided that in a neighborhood G of x_0 we use $K_1 \exp[mF(x)/\sqrt{\epsilon}]$ (for $K_1 = \max_{G\cap\Gamma} |\varphi(x)|$ and $m > 0$ appropriately chosen) as the boundary layer term. The validity of this substitution follows essentially from the proof of Theorem 4.3.

Remark 4.3. Often the estimate of Theorem 4.5 can be strengthened in a manner analogous to the strengthening of Theorem 2.5 in Section 3. Namely, in a neighborhood G of a point x_0 on Γ in which $B(x,u) = \sum_{j=1}^{N} b_j(x,u) F_{x_j}^2(x)$ changes sign for all u of interest we can give sharp estimates for the boundary layer term. These estimates depend of course on

the sign of B relative to the sign of the boundary data φ, and they are the analogs of the estimates in Lemmas 3.1 and 3.2. We leave their precise formulation to the reader.

Part II

Robin Problems

5. The Problem $(\mathcal{R}_3)$. We turn now to a discussion of results for the Robin problem

$$(\mathcal{R}_3)\qquad \begin{aligned} &\epsilon\Delta u = \mu A(x,u)\cdot\nabla u + h(x,u)\,,\ x \text{ in } \Omega = \{x : F(x) < 0\}\,,\\ &\beta(x)u + \nabla F(x)\cdot\nabla u = \varphi(x)\,,\ x \text{ on } \Gamma = \partial\Omega\,, \end{aligned}$$

which are similar to those for the corresponding Dirichlet problem $(\mathcal{R}_1)$. The function β is assumed to be nonnegative on Γ and of class $C^{(2,\alpha)}(\Gamma_\delta)$ while the functions A, h, F and φ are as above. The only exception is that we assume here and in the next section that $\|\nabla F(x)\| \equiv 1$ for x on $\Gamma = F^{-1}(0)$, that is, in particular, Γ is not allowed to have corners. Thus the exterior unit normal at a point x on Γ, $\hat{n}(x)$, is equal to $\nabla F(x)/\|\nabla F(x)\| = \nabla F(x)$, and so $\nabla F(x)\cdot\nabla u = \hat{n}(x)\cdot\nabla u = \frac{\partial u}{\partial n}$ is the outward normal derivative. In view of this the boundary condition in $(\mathcal{R}_3)$ can be written in the more familiar form $\beta u + \frac{\partial u}{\partial n} = \varphi$.

We assume as usual that $h(x,0) \equiv 0$ in Ω; however, the non-Dirichlet boundary condition prompts us to seek solutions of $(\mathcal{R}_3)$ which are uniformly close to zero in $\bar{\Omega}$ (cf. for example [13]). Thus we will look for solutions of $(\mathcal{R}_3)$ in the domain $\mathcal{E} = \bar{\Omega} \times \{u : |u| \le \delta\}$ in $\bar{\Omega} \times \mathbb{R}^1$ and we ask further that the zero solution be stable in the sense that there exist a

positive constant m such that $h_u(x,u) \geq m^2 > 0$ for (x,u) in $\mathcal{E}$. Note that since $\delta > 0$ is arbitrarily small (but independent of ϵ and μ) this stability assumption is essentially that $h_u(x,0) > 0$ in $\overline{\Omega}$; this condition is much less restrictive than the corresponding condition for the Dirichlet problems $(\mathcal{N}_1)$ and $(\mathcal{N}_2)$.

Our results for the problem $(\mathcal{N}_3)$ are so similar to those for the problem $(\mathcal{N}_1)$ that we will only prove the first of the two theorems stated below. Moreover, for ease of presentation we will state these results in abbreviated form, it being understood that the domains $\mathcal{D}$ and $\mathcal{D}_\delta$ are replaced by the domains $\mathcal{E}$ and $\mathcal{E}_\delta$ respectively and that the definition of stability is that given in this section. (Here $\mathcal{E}_\delta = \mathcal{E} \cap (\Gamma_\delta \times \mathbb{R}^1)$.)

We consider first the situation in which the μ- term does not really affect the behavior of solutions.

Theorem 5.1. The conclusions of Theorem 2.1, (Theorem 2.2) and Theorem 2.3 are valid for the problem $(\mathcal{N}_3)$ with K replaced by $\sqrt{\epsilon}\, K m_1^{-1} (\sqrt{\epsilon}\, K m_2^{-1})$.

Proof. The theorems for Robin problems will be proved by using a variation of the differential inequality theorem quoted in Section 2 and due to Amann [1]. Namely, for $\mathcal{F}(x, u, \nabla u, \mu) = \mu A(x,u) \cdot \nabla u + h(x,u)$ or $\sum_{j=1}^{N} b_j(x,u) u_{x_j}^2 + h(x,u)$ satisfying the given smoothness conditions in $\mathcal{E}$ the boundary value problem $\epsilon \Delta u = \mathcal{F}(x, u, \nabla u, \mu)$, x in Ω, $\beta u + \frac{\partial u}{\partial n} = \varphi(x)$, x on $\Gamma = \partial\Omega$, with $\beta \geq 0$, has a solution $u = u(x; \epsilon, \mu)$ in $\mathcal{E}$ of class $C^{(2,\alpha)}(\overline{\Omega})$ for each $\epsilon > 0$ and $\mu > 0$ for which there exist boundary-related functions $\underline{\omega}, \overline{\omega}$ of class $C^{(2,\alpha)}(\overline{\Omega})$ such that $\underline{\omega} \leq \overline{\omega}$, $\beta\underline{\omega} + \partial\underline{\omega}/\partial n\,|_\Gamma \leq \varphi \leq \beta\overline{\omega} + \partial\overline{\omega}/\partial n\,|_\Gamma$, and $\epsilon\Delta\underline{\omega} \geq \mathcal{F}(x, \underline{\omega}, \nabla\underline{\omega}, \mu)$, $\epsilon\Delta\overline{\omega} \leq \mathcal{F}(x, \overline{\omega}, \nabla\overline{\omega}, \mu)$ for x in Ω. Moreover, this solution satisfies $\underline{\omega}(x; \epsilon, \mu) \leq u(x; \epsilon, \mu) \leq \overline{\omega}(x; \epsilon, \mu)$ for x in $\overline{\Omega}$. The bounding functions $\underline{\omega}, \overline{\omega}$ are said

to be boundary-related if there exists a function $\tilde{u} = \tilde{u}(x\,;\epsilon\,,\mu)$ of class $C^{(2,\alpha)}(\bar{\Omega})$ such that for x on Γ $\beta(x)\tilde{u} + \frac{\partial\tilde{u}}{\partial n} = \varphi(x)$ and for x in $\bar{\Omega}$ $\underline{\omega}(x\,;\epsilon\,,\mu) \leq \tilde{u}(x\,;\epsilon\,,\mu) \leq \underline{\omega}(x\,;\epsilon\,,\mu)$.

Turning now to the proof of the theorem consider just the case $u << \sqrt{\epsilon}$, as the other two cases follow similarly. Define for x in $\bar{\Omega}$ the functions $\underline{\omega}, \bar{\omega}$ by $\underline{\omega}(x,\epsilon) = -\bar{\omega}(x,\epsilon) = -\sqrt{\epsilon}\,Km_1^{-1}\exp[m_1 F(x)/\sqrt{\epsilon}\,]$. Clearly $\underline{\omega} \leq \bar{\omega}$ and $\beta\underline{\omega} + \nabla F\cdot\nabla\underline{\omega}\,|_\Gamma \leq \varphi \leq \beta\bar{\omega} + \nabla F\cdot\nabla\bar{\omega}\,|_\Gamma$. (Recall that $\beta \geq 0$.) In addition, we have that (for $\mathfrak{F}(x\,,u\,,\nabla u\,,\mu) = \mu A(x,u)\cdot\nabla u + h(x,u)$)

$$\begin{aligned}\epsilon\Delta\underline{\omega} - \mathfrak{F}(x\,,\underline{\omega}\,,\nabla\underline{\omega}\,,\mu) &\geq \{-\,|\,\Delta F\,|_\infty\,\epsilon - m_1\sqrt{\epsilon}\ \ \|\nabla F\|^2 \\ &\quad + \mu A\cdot\nabla F + m^2 m_1^{-1}\sqrt{\epsilon}\,\}\,(m_1/\sqrt{\epsilon}\,)\,|\underline{\omega}| \\ &\geq 0 \text{ by our choice of } m_1\end{aligned}$$

if ϵ is sufficiently small, say $0 < \epsilon \leq \epsilon_0$. Similarly we can show that $\epsilon\Delta\bar{\omega} \leq \mathfrak{F}(x\,,\bar{\omega}\,,\nabla\bar{\omega}\,,\mu)$ in Ω , and so the conclusion of the theorem will follow once we establish that our functions $\underline{\omega}, \bar{\omega}$ are also boundary-related. To this end, let $\tilde{\varphi}$ be a $C^{(2,\alpha)}$-extension of φ to $\bar{\Omega}$, that is, $\tilde{\varphi}|_\Gamma = \varphi$, and define the function $\tilde{u}$ by $\tilde{u}(x,\epsilon) = \tau(\epsilon)\,\tilde{u}(x)\,(\,\exp[\,F(x)/\tau(\epsilon)\,] - 1)$, where $\tau(\epsilon) > 0$ is a transcendentally small term. Then clearly we have that $\underline{\omega} \leq \tilde{u} \leq \bar{\omega}$ in $\bar{\Omega}$ and $\beta\tilde{u} + \nabla F\cdot\nabla\tilde{u}\,|_\Gamma = \beta\cdot 0 + \tilde{\varphi}\,|_\Gamma = \varphi$, and so indeed $\underline{\omega}, \bar{\omega}$ are boundary-related. (Recall that $\|\nabla F\| \equiv 1$ on Γ .)

If now the μ-term is sufficiently large and either the inequality (**) obtains or the inequality (*) does not obtain then we have the following result.

<u>Theorem 5.2</u>. <u>The conclusion of Theorem 2.4 (Theorem 2.5) is valid for the problem</u> $(\mathcal{N}_3)$ <u>with</u> K <u>replaced by</u> $(\epsilon/\mu)\,Kk^{-1}(\mu\,Km_3^{-1})$.

We close this section with two remarks.

Remark 5.1. The estimate in the conclusion of Theorem 5.2 for the case $\sqrt{\epsilon} << \mu$ and $A(x,u) \cdot \nabla F(x) \not\geq 0$ in $\mathcal{E}_\delta$ can be sharpened in the same way that Lemmas 3.1 and 3.2 sharpen the estimate in Theorem 2.5. Namely, it is not difficult to see that the conclusions of Lemmas 3.1 and 3.2 are valid for the problem $(\mathcal{P}_3)$ with K_1 replaced by $\nu K_1 k^{-1}$.

Remark 5.2. The Robin problem $(\mathcal{P}_3)$ can be studied under definitions of stability which are more general than the one used here. The interested reader should consult [13] where a related class of problems is treated.

6. The Problem $(\mathcal{P}_4)$. We consider finally the problem

$$(\mathcal{P}_4)\quad \begin{aligned} &\epsilon\Delta u = \mu \sum_{j=1}^{N} b_j(x,u)\, u_{x_j}^2 + h(x,u)\ , \ x \ \text{in} \ \Omega = \{x : F(x) < 0\}\ , \\ &\beta(x)u + \nabla F(x) \cdot \nabla u = \varphi(x)\ , \ x \ \text{on} \ \Gamma\ , \end{aligned}$$

where all of the functions appearing here are as defined in Sections 4 and 5. The results for $(\mathcal{P}_4)$ analogous to those for the Dirichlet problem $(\mathcal{P}_2)$ follow directly once the appropriate changes are made in order to accommodate the non-Dirichlet boundary conditions.

We assume then that $h(x,0) \equiv 0$ in Ω and that there exists a positive constant m such that $h_u(x,u) \geq m^2 > 0$ for (x,u) in the domain $\mathcal{E}$ defined in Section 5. We will only prove the first of the next three theorems since the arguments for the other two are quite similar.

The first theorem deals with the case in which the μ - term does not really affect the behavior of solutions.

<u>Theorem 6.1</u>. <u>The conclusions of Theorem 4.1, (Theorem 4.2) and Theorem 4.3 are valid for the problem</u> (η_4) <u>with</u> K <u>replaced by</u> $\sqrt{\epsilon}\, K m_1^{-1}$ $(\sqrt{\epsilon}\, K m_2^{-1})$.

Note that domains $\mathcal{D}$ and $\mathcal{D}_\delta$ of Section 4 are replaced by the domains $\mathcal{E}$ and $\mathcal{E}_\delta$ respectively.

<u>Proof</u>. We will prove this result only in the case that $\mu << \sqrt{\epsilon}$ since the proofs in the other two cases are very similar. Define for x in $\overline{\Omega}$ the functions $\underline{\omega}, \overline{\omega}$ by

$$\underline{\omega}(x,\epsilon) = -\overline{\omega}(x,\epsilon) = -\sqrt{\epsilon}\, K m_1^{-1} \exp[\, m_1 F(x)/\sqrt{\epsilon}\,] .$$

Then $\underline{\omega} \leq \overline{\omega}$, $\beta\underline{\omega} + \nabla F \cdot \nabla\underline{\omega}|_\Gamma \leq \varphi \leq \beta\overline{\omega} + \nabla F \cdot \nabla\overline{\omega}|_\Gamma$, and for $\mathcal{F}(x,u,\nabla u,\mu) = \mu \sum_{j=1}^{N} b_j(x,u)\, u_{x_j}^2 + h(x,u)$ we have that

$$\begin{aligned}
\epsilon\Delta\underline{\omega} - \mathcal{F}(x,\underline{\omega},\nabla\underline{\omega},\mu) \geq \{ &-\epsilon\,|\Delta F|_\infty - \sqrt{\epsilon}\; m_1 L^2 \\
&- m_1(\mu/\sqrt{\epsilon})\,\Big(\sum_{j=1}^{N} b_j(x,\underline{\omega})\, F_{x_j}^2\Big)\,|\underline{\omega}| \\
&+ \sqrt{\epsilon}\; m_2 m_1^{-1}\}(m_1/\sqrt{\epsilon})\,|\underline{\omega}| \\
&\geq 0 \text{ by our choice of } m_1
\end{aligned}$$

if ϵ is sufficiently small, say $0 < \epsilon \leq \epsilon_0$. Similarly, we see that $\epsilon\Delta\overline{\omega} - \mathcal{F}(x,\overline{\omega},\nabla\overline{\omega},\mu) \leq 0$ in Ω. The conclusion of the theorem now follows from Amann's theorem quoted in Section 5 once we observe that the verification of the boundary-relatedness of $\underline{\omega}, \overline{\omega}$ follows exactly as in the proof of Theorem 5.1.

We consider next the case in which $\sqrt{\epsilon} << \mu$ and the strong inequality (**) of Section 4 obtains in the domain $\mathcal{E}_\delta$.

Theorem 6.2. *Suppose that* $\sqrt{\epsilon} << \mu$ *and that the inequality* $(**)$ *of Section* 4 *obtains in* $\mathcal{E}_\delta$. *Then the conclusion of Theorem* 4.4 *is valid for the problem* $(\mathcal{N}_4)$ *with the* ℓn*-term replaced by*

$$\nu \ell n \{ \ell (K^{-1} - kF(x)\nu^{-1})^{-1} \} ,$$

where ℓ *is such that* $\ell K^{-1} > 1$ *and* $\nu = \epsilon/\mu$.

Suppose finally that $\sqrt{\epsilon} << \mu$ but that the inequality $(*)$ of Section 4 does not obtain in $\mathcal{E}_\delta$, then there is the following result.

Theorem 6.3. *Suppose that* $\sqrt{\epsilon} << \mu$ *and that* $\varphi(x) \sum_{j=1}^{N} b_j(x,u) F^2_{x_j}(x) \not\geq 0$ *for* (x,u) *in* $\mathcal{E}_\delta$. *Then the conclusion of Theorem* 4.5 *is valid for the problem* $(\mathcal{N}_4)$ *with the exponential term replaced by* $\mu K m_1^{-1} \exp[m_1 F(x)/\mu]$.

We close with the observation that it is possible to consider Robin problems for the more general equation $\epsilon \Delta u = \mu_1 \sum_{j=1}^{N} b_j(x,u) u^2_{x_j} + \mu_2 A(x,u) \cdot \nabla u + h(x,u)$. Moreover, the estimate of Theorem 6.3 can be improved by closer analysis of the type used in Section 3 (cf. also Remark 5.1).

Part III

Examples

7. Some Examples. In this section we discuss several examples which illustrate the theory developed above.

Example 7.1. Consider first the Dirichlet problem

$$\epsilon \Delta u = \epsilon^{\sigma} A(x,u) \cdot \nabla u + u , x \text{ in } \Omega = \{x : F(x) < 0\} ,$$

(E1)

$$u(x,\epsilon) = \varphi(x) , \; x \text{ on } \Gamma = \partial\Omega ,$$

where σ is a positive constant and the functions A, F and φ have the usual smoothness properties. This problem has a solution $u = u(x,\epsilon)$ which is unique by the maximum principle (cf. [19]) for each $\epsilon > 0$ sufficiently small since the functions $\underline{\omega}, \overline{\omega}$ defined by $\underline{\omega} = -\overline{\omega} = -K$ for $K = \max_{\Gamma} |\varphi(x)|$ are bounding functions. Moreover, for x in $\overline{\Omega}$ we have the (rather crude) estimate $|u(x,\epsilon)| \leq K$. In order to obtain sharper estimates we apply the theory of Section 2.

Suppose first that the region Ω is the unit ball in $\mathbb{R}^N$ centered at 0, that is, $F(x) = \frac{1}{2}(\|x\|^2 - 1)$. If $\sigma \geq \frac{1}{2}$ then the specific form of A is immaterial and Theorem 2.1 or 2.2 shows that the solution of (E1) satisfies in $\overline{\Omega}$

$$(7.1) \qquad u(x,\epsilon) = \mathcal{O}(K \exp[F(x)/\sqrt{\epsilon}]) .$$

However, if $\sigma < \frac{1}{2}$ then we know that it is the interaction of A with Γ which determines the width of the boundary layer along Γ. For example, if $A(x,u) = u^2 x$ then $A(x,u) \cdot \nabla F(x) = u^2 \|x\|^2 \geq 0$, and so Theorem 2.3 implies that the solution of (E1) also satisfies the estimate (7.1) for this range of σ. Similarly if $A(x,u) = kx$ for a positive constant k then $A(x,u) \cdot \nabla F(x) = k\|x\|^2$ and we deduce from Theorem 2.4 that the solution u satisfies the sharper estimate in $\overline{\Omega}$

$$(7.2) \qquad u(x,\epsilon) = \mathcal{O}(K \exp[kF(x)/\epsilon^{1-\sigma}]) .$$

Finally if $A(x,u)$ is such that $A(x,u) \cdot x \not\geq 0$ in $\mathcal{D}_\delta$ then the most general estimate we can give for u is that of Theorem 2.5, namely for x in $\overline{\Omega}$

$$u(x,\epsilon) = \mathcal{O}(K \exp[F(x)/\epsilon^{\sigma}]) .$$

For such an A it may be possible to give sharper estimates like those in Lemmas 3.1 and 3.2 along the portions of Γ where $A(x,u)\cdot x \geq 0$ or $A(x,u)\cdot x > 0$.

Suppose finally that Ω is the unit cube in $\mathbb{R}^N$ centered at 0, that is, $F(x) = -\prod_{j=1}^{N}(1-x_j^2)$ for $|x_j| \leq 1$. Then $\Gamma = \partial\Omega = F^{-1}(0)$ has 2^N vertices or corners $\{\xi_r\}$ and $\nabla F(\xi_r) = 0$ for $1 \leq r \leq 2^N$. Nevertheless if, for example, $A(x,u) = u^2 x$ then $A(x,u)\cdot\nabla F(x) =$

$$u^2 x \cdot 2\Big(x_1 \prod_{j=2}^{N}(1-x_j^2),\ x_2 \prod_{\substack{j=1\\ j\neq 2}}^{N}(1-x_j^2),\ \dots,\ x_N \prod_{j=1}^{N-1}(1-x_j^2)\Big) =$$

$$= 2u^2\Big\{x_1^2 \prod_{j=2}^{N}(1-x_j^2) + x_2^2 \prod_{\substack{j=1\\ j\neq 2}}^{N}(1-x_j^2) + \dots + x_N^2 \prod_{j=1}^{N-1}(1-x_j^2)\Big\} \geq 0$$

and so for all $\sigma > 0$ the estimate (7.1) is valid. While if $A(x,u) = x$ and $\sigma < \frac{1}{2}$ then $A(x,u)\cdot\nabla F(x) \geq \frac{1}{2}\|\nabla F\|^2$ in $\overline{\Omega}$ and so the estimate (7.2) is valid with $k = \frac{1}{2}$.

Example 7.2. Consider now the Robin problem

$$\text{(E2)}\qquad \begin{aligned} &\varepsilon\Delta u = \varepsilon^{\sigma} A(x,u)\cdot\nabla u + u^3 - u,\ x \text{ in } \Omega, \\ &u + \frac{\partial u}{\partial n} = \varphi(x),\ x \text{ on } \Gamma, \end{aligned}$$

where σ is again a positive constant and Ω is the unit ball in $\mathbb{R}^N$ centered at 0; consequently, $\frac{\partial u}{\partial n} = x\cdot\nabla u$ is the outward normal derivative of u. Suppose that the function A satisfies our smoothness assumptions. We note first that the reduced equation $h(u) = u^3 - u = 0$ has the three solutions $0, 1$ and -1; of these, ± 1 are stable while 0 is unstable in the sense that $h'(0) = -1 < 0$. The theory of Section 5 applies to the stable zero 1 (-1) if we make the change of variable $\tilde{u} = u - 1$ $(\tilde{u} = u + 1)$, and so we will simply quote a theorem from this section with the tacit understanding that the appropriate change of variable has been made.

If $\sigma \geq \frac{1}{2}$ or if $\sigma < \frac{1}{2}$ and $A(x,u) \cdot \nabla F(x) = A(x,u) \cdot x \geq 0$ in $\mathcal{E}_\delta^+ = \Gamma_\delta \times \{u : |u-1| \leq \delta\}$ then Theorem 5.1 implies that the problem (E2) has a solution $u = u_+(x,\epsilon)$ such that in $\overline{\Omega}$

$$(7.3^+) \qquad u_+(x,\epsilon) = 1 + \mathcal{O}(\sqrt{\epsilon/2}\ K^+ \exp[\sqrt{2}\, F(x)/\sqrt{\epsilon}\,])$$

for $K^+ = \max_\Gamma |\varphi(x) - 1|$. Similarly, for these choices of σ and for $A(x,u) \cdot x \geq 0$ in $\mathcal{E}_\delta^- = \Gamma_\delta \times \{u : |u+1| \leq \delta\}$ if $\sigma < \frac{1}{2}$ this theorem implies the existence of a second solution $u = u_-(x,\epsilon)$ such that in $\overline{\Omega}$

$$(7.3^-) \qquad u_-(x,\epsilon) = -1 + \mathcal{O}(\sqrt{3/2}\ K^- \exp[\sqrt{2}\, F(x)/\sqrt{\epsilon}\,])$$

for $K^- = \max_\Gamma |\varphi(x) + 1|$.

Suppose next that $\sigma < \frac{1}{2}$. If A is such that there exists a positive constant k for which $A(x,u) \cdot x \geq k\,\|x\|^2$ in $\mathcal{E}_\delta^+$ and/or $\mathcal{E}_\delta^-$ then Theorem 5.2 implies that the problem (E2) has a solution u_+ and/or a solution u_- such that in $\overline{\Omega}$

$$(7.4^\pm) \qquad u_\pm(x,\epsilon) = \pm 1 + \mathcal{O}(k^{-1} \epsilon^{1-\sigma} K^\pm \exp[\,k\,F(x)/\epsilon^{1-\sigma}\,]) .$$

Finally, if $A(x,u) \cdot x \ngeq 0$ in $\mathcal{E}_\delta^+$ and/or $\mathcal{E}_\delta^-$ then there are still two solutions u_+ and u_- of (E2); however, the best general estimate we can give is

$$u_\pm(x,\epsilon) = \pm 1 + \mathcal{O}(\epsilon^\sigma K^\pm \exp[\,F(x)/\epsilon^\sigma\,]) .$$

A closer analysis may allow us to give sharper estimates of the form $(7.3^\pm)$ or $(7.4^\pm)$ for this case if $A(x,u) \cdot x \geq 0$ along portions of Γ (cf. Remark 5.1).

Before passing to another example we note that a similar equation can be used to illustrate Remark 2.2. Namely, consider the Dirichlet problem

$$\varepsilon \Delta u = \varepsilon^{\sigma} A(x,u) \cdot \nabla u + u - u^3 , \quad x \text{ in } \Omega ,$$

(E2′)

$$u(x,\varepsilon) = \varphi(x) , \quad x \text{ on } \Gamma .$$

Then the reduced equation $\tilde{h}(u) = u - u^3 = 0$ has 0 as its only stable solution and $\tilde{h}'(u) = 1 - 3u^2 > 0$ provided $|u| < 1/\sqrt{3}$. Thus for all boundary values φ such that $\max_{\Gamma} |\varphi(x)| < 1/\sqrt{3}$ we can apply the theory of Section 2. However this restriction on φ can be weakened if we use the integral condition (ii) in Remark 2.2, that is,

$$\varphi(x) \int_0^{\eta} (s - s^3)\, ds > 0 \quad \text{for all } \eta \text{ in } (0,\varphi(x)] \text{ or } [\varphi(x),0) .$$

For instance, if $\varphi(x) > 0$ then $\int_0^{\eta} (s - s^3)\, ds = \frac{\eta^2}{4}(2 - \eta^2) > 0$ provided $0 < \eta < \sqrt{2}$, with a similar result for $\varphi(x) < 0$. Consequently for those φ such that $\max_{\Gamma} |\varphi(x)| < \sqrt{2}$ we can actually apply the theory of Section 2.

Example 7.3. In order to illustrate the theory for the quadratic problems $(\mathcal{N}_2)$ and $(\mathcal{N}_4)$ we consider an equation which models a first-order gaseous reaction that occurs inside of a porous catalyst pellet, namely $P \rightarrow nQ$, where n is the stoichiometric factor. If u is the normalized concentration of P and $\theta(\sim n - 1)$ is the volume change modulus then a possible conservation law governing this reaction is (cf. [2]).

$$\nabla \cdot [(1 + \theta u)^{-1} \nabla u] = \Phi^2 u , \quad x \text{ in } \Omega . \tag{7.5}$$

Here Φ^2 is the Thiele modulus which measures the relative importance of reaction versus diffusion and $\Omega \subset \mathbb{R}^3$ is the idealized region in which the reaction takes place. Suppose further that the reaction produces an increase in volume (that is, $\theta > 0$) and that the reaction is diffusion-limited (that is, $\Phi^2 >> 1$). Then equation (7.5) can be cast into the more familiar form

$$\epsilon\Delta u = \epsilon\theta(1+\theta u)^{-1}\|\nabla u\|^2 + u(1+\theta u),$$

where $\epsilon = \Phi^{-2} << 1$.

If we assume that a unit concentration of P is prescribed on the surface Γ of the region Ω then the problem to be solved is one of the form $(\mathcal{P}_2)$, namely

$$\text{(E 3)}\quad \begin{aligned} &\epsilon\Delta u = \epsilon\theta(1+\theta u)^{-1}\|\nabla u\|^2 + u(1+\theta u),\ x \text{ in } \Omega,\\ &u(x,\epsilon) \equiv 1,\ x \text{ on } \Gamma. \end{aligned}$$

For $\mu = \epsilon\theta$, $\mu = \mathcal{O}(\epsilon) << \sqrt{\epsilon}$ since $\theta = \mathcal{O}(1)$, and $\bar{h}(u) = u(1+\theta u)$ satisfies $\bar{h}'(u) \geq 1$ for $u \geq 0$. (Since u measures concentration we are only interested in nonnegative solutions of (E 3).) Therefore Theorem 4.1 implies that the problem (E 3) has a solution $u = u(x,\epsilon)$ for each $\epsilon > 0$ such that in $\bar{\Omega}$

$$\text{(7.6)}\qquad 0 \leq u(x,\epsilon) \leq \exp[F(x)/\sqrt{\epsilon}],\ \text{where } \Gamma = F^{-1}(0).$$

The assumption that $u \equiv 1$ on Γ is tantamount to the assumption that there is no resistance to the transfer of reactant from the bulk flow to the surface of the pellet. A more realistic assumption is that there is in fact a small amount of resistance, and so we could consider the more general problem

$$\text{(E 3')}\quad \begin{aligned} &\epsilon\Delta u = \epsilon\theta(1+\theta u)^{-1}\|\nabla u\|^2 + u(1+\theta u),\ x \text{ in } \Omega,\\ &u + (\mathcal{S}h)^{-1}\frac{\partial u}{\partial n} = 1,\ x \text{ on } \Gamma. \end{aligned}$$

Here $\mathcal{S}h$ is the resistance coefficient which is known as the Sherwood number. This time Theorem 6.1 implies that the problem (E3') has a solution $u = u(x,\epsilon)$ for each $\epsilon > 0$ such that in $\bar{\Omega}$

$$\text{(7.7)}\qquad 0 \leq u(x,\epsilon) \leq \sqrt{\epsilon}\ \mathcal{S}h \exp[F(x)/\sqrt{\epsilon}].$$

Note that if $Sh = \mathcal{O}(1/\sqrt{\epsilon}) >> 1$ then the resistance to transfer is negligible and the estimate (7.7) reduces to (7.6).

The estimates (7.6) and (7.7) show that the assumption of diffusion-limitation ($\Phi^2 >> 1$ or $\epsilon << 1$) implies that most of the reaction is confined to the surface of the pellet. This phenomenon has been observed experimentally (cf. [2]).

Example 7.4. As our final example we consider the linear problem

$$\text{(E4)} \qquad \begin{aligned} &\epsilon \Delta u = \mu u_x + u \,, \; (x,y) \text{ in } \Omega = \{(x,y) : F(x,y) < 0\} \,, \\ &u(x,y\,;\epsilon,\mu) = \varphi(x,y) \,, \; (x,y) \text{ on } \Gamma \,, \end{aligned}$$

where Ω is the square of length 2 centered at $(0,0)$, that is, $F(x,y) = -(1-x^2)(1-y^2)$. This problem for $\sqrt{\epsilon} << \mu$ was studied by O'Malley [18; Sec. 7] who gave estimates for the solution u in the horizontal strip $\{(x,y) : -1 \leq x \leq 1, \; -1 < -1 + \delta \leq y \leq 1 - \delta < 1\}$. He was unable however to analyze u near the corners of Ω and along the sides $y = \pm 1$ since these straight lines are the characteristic curves of the semi-reduced equation $\mu u_x + u = 0$. Using our theory we can discuss the behavior of solutions of (E4) throughout $\overline{\Omega}$.

First of all, the problem (E4) has a unique solution for all $\epsilon > 0$ and $\mu > 0$ since the functions $\underline{\omega}, \overline{\omega}$ defined by $\underline{\omega} = -\overline{\omega} = -K$ for $K = \max_{\Gamma} |\varphi(x,y)|$ are bounding functions and since the equation satisfies a maximum principle. If $\mu << \sqrt{\epsilon}$ or $\mu \sim \sqrt{\epsilon}$ then Theorems 2.1 and 2.2 imply that this solution satisfies the estimate in $\overline{\Omega}$

$$u(x,y\,;\epsilon,\mu) = \mathcal{O}(K \exp[\,F(x,y)/\sqrt{\epsilon}\,]) \,.$$

Suppose however that $\sqrt{\epsilon} << \mu$; then we must examine the function $\gamma(x,y,u) = (1,0) \cdot \nabla F = 2x(1-y^2)$. Since $\gamma \not\geq 0$ for (x,y) in $[-1,1] \times [-1,1]$ the most general estimate in $\bar{\Omega}$ we can give for u is (cf. Theorem 2.5)

$$u(x,y;\epsilon,\mu) = \mathcal{O}(K \exp[F(x,y)/\mu]) .$$

Now let us apply the results of Section 3. First, we look in a neighborhood of the right hand boundary $x = 1$. For all y in $[-1+\delta, 1-\delta]$ (for $\delta > 0$ small and independent of ϵ and μ) a calculation shows that there is a positive constant $k = k(\delta)$ such that $\gamma \geq k \|\nabla F\|^2$, and so Lemma 3.1 implies that for such x and y

$$u(x,y;\epsilon,\mu) = \mathcal{O}(K \exp[kF(x,y)\mu/\epsilon]) .$$

This estimate agrees with that of O'Malley. For those values of y in $[-1,-1+\delta)$ and $(1-\delta,1]$ we still have that $\gamma \geq 0$ and so this lemma implies that the solution u satisfies there the estimate

$$u(x,y;\epsilon,\mu) = \mathcal{O}(K \exp[F(x,y)/\nu]) , \tag{7.8}$$

where $\nu = \sqrt{\epsilon}$ for y such that $1-y^2 = \mathcal{O}(\sqrt{\epsilon}/\mu)$ and $\nu = \mathcal{O}(\epsilon/((1-y^2)\mu))$ for y such that $1-y^2 >> \sqrt{\epsilon}/\mu$. Secondly, we look in a neighborhood of the left hand boundary $x = -1$. Here $\gamma \leq 0$ and so Lemma 3.2 implies that the solution u satisfies the estimate

$$u(x,y;\epsilon,\mu) = \mathcal{O}(K \exp[F(x,y)/\nu]) \tag{7.9}$$

where $\nu = \sqrt{\epsilon}$ for y such that $1-y^2 = \mathcal{O}(\sqrt{\epsilon}/\mu)$ and $\nu = \mathcal{O}((1-y^2)\mu)$ for y such that $1-y^2 >> \sqrt{\epsilon}/\mu$. Note that for y in $[-1+\delta, 1-\delta]$

the estimate (7.8) becomes

$$u(x,y\,;\epsilon,\mu) = \mathcal{O}(\,K\exp[\,F(x,y)/\mu\,]\,)\ ;$$

this result agrees with that of O'Malley.

It remains to consider the upper and lower boundaries of Ω. We can restrict our discussion to the upper boundary $y = 1$ and note that our results apply to the lower boundary by reflection. Let us look then in a neighborhood of $y = 1$. For $x \geq 0$ the function γ is nonnegative and so the estimate (7.8) applies to u for such x and y; similarly, for $x \leq 0$ we have that $\gamma \leq 0$ and so the estimate (7.9) is valid for these x and y.

Finally we remark that in a neighborhood of the corner points $(-1, \pm 1)$ (where $\gamma \leq 0$) the estimates (7.9) hold, while in a neighborhood of the corner points $(1, \pm 1)$ (where $\gamma \geq 0$) the estimates (7.8) are valid.

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This problem determines the asymptotic solution of boundary value problems for nonlinear singularly perturbed equations such as

$$\varepsilon\Delta u = \mu A(x,u)\cdot\nabla u + h(x,u)$$

where ε and μ are small parameters. The equations are more nonlinear than those considered previously, and more detailed information is obtained about boundary layer behavior and the necessary interrelationships between parameters.

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